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On real closed rings of higher level

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Abstract

The purpose of this paper is to introduce the notion of real closed rings of higher level in the category of reduced commutative rings. Such real closed rings of higher level are just generalized real closed fields in case the rings considered are fields. With the aid of the theory of totally integrally closed rings, some important results on real closed rings of higher level are established. The results in this paper involve real valuations and a related characterization of real closed rings of higher level.

Keywords: Ordering of higher level; Real closed ring of higher level; Real valuation; Tight integral extension; Totally integral closure

0. Introduction

In the theory of real fields, the notion of real closed fields plays an important role, see [16]. As a generalization of the notion of real fields, the notions of semireal rings and real rings have been introduced in the category of commutative rings. The study of semireal rings and real rings has become an area of the subject called real algebra (see [12]), and many important results on the two kinds of rings have been established and applied. The notion of real closed fields have been generalized to the category of commutative rings in many manners. In Ref. [17], the notion of real closed rings is introduced by means of tight totally integrally closed rings. Recently, in Ref. [19] we introduce the so-called semireal closed rings, which have more desirable properties than real closed rings introduced in [17].

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In case the rings considered are fields, both real closed rings and semireal closed rings coincide with real closed fields.

As another successful generalization of the Artin–Schreier theory, E. Becker first introduced the notion of orderings of higher level for a formally real field. Many important results on orderings of higher level may be found in Refs. [2–7]. In the theory of higher level orderings, the role played by real closed fields is replaced by real closed fields of higher level (i.e., generalized real closed fields). It is a natural trend to extend the theory of higher level orderings from the category of fields to that of commutative rings, and many important results on higher level orderings of fields have been generalized to commutative rings, e.g., see Refs. [15,18]. It is a natural question how to generalize reasonably the notion of real closed fields of higher level to the category of commutative rings.

The purpose of this paper is to introduce the notion of real closed rings of higher level. Such real closed rings are exactly real closed fields of higher level in case the rings considered are fields. The results in this paper show that our notion of real closed rings is reasonable.

Throughout this paper, "ring" means "commutative ring with identity 1," and every subring has the same identity as its overring. A ring R is called semireal if -1 is not a sum of squares in R. For a ring R, an extension ring S of R is said to be tight if every non-zero ideal of S contracts to a non-zero ideal of R. In Ref. [10], Enochs introduced the concept of a totally integrally closed ring to generalize the concept of an algebraically closed field. A ring I is called a totally integrally closed ring if, for any ring homomorphism $\sigma : A \to I$ and any integral extension B of A, there is a homomorphism of B into I extending σ. In order to apply the theory of totally integrally closed rings, the rings considered in this paper are always assumed to be reduced rings, i.e., rings without non-zero nilpotent element. According to Theorem 2 in [10], every ring R considered in this paper possesses a tight integral extension, called the totally integral closure of R, which is totally integrally closed and is also unique up to R-isomorphism. Some concepts and results related to totally integrally closed rings can be found in [10,11].

Moreover, in what follows, the following symbols are kept: $\mathbb{N} :=$ the set of positive integers; $\mathbb{Z} :=$ the ring of integers. For any subset S of a ring R, $\dot{S}$ stands for the set of all non-zero elements in S. Moreover, for two sets A, B, write $A \setminus B$ for the set $\{x \mid x \in A \text{ but } x \notin B\}$.

1. Real closed rings of higher level

In this section, we introduce the notion of real closed rings of higher level, and establish some important facts about such real closed rings. First, let us recall the definitions of preorderings and orderings of higher level of a ring. Let R be a ring. A subset T of R is called a preordering of level n if $T + T \subseteq T$, $T \cdot T \subseteq T$, $-1 \notin T$, and $x^{2n} \in T$ for every $x \in R$. A preordering P of level n of R is called an ordering of level n, if the additional conditions are satisfied: (1) $\wp := P \cap -P$ is a prime ideal of R; (2) for $x, y \in R$, $xy^{2n} \in P$ implies $x \in P$ or $y \in \wp$; and (3) $\widetilde{P} := \{\tilde{p}\tilde{x}^{-2n} \mid p \in P, \text{ and } x \in R \setminus \wp\}$ is an ordering of level n of the fraction field $k(\wp)$ of $R/\wp$, where $\tilde{x}$ is the image of x under the canonical homomorphism of R into $R/\wp$ for $x \in R$. If the ordering $\widetilde{P}$ is of exact level m, i.e., the quotient group $(k(\wp) \setminus \{0\})/(\widetilde{P} \setminus \{0\})$ is a cyclic group of order $2m$, we shall call P an ordering of exact level m of R. In this case, m is obviously a divisor of n. When P is an ordering of level n of R, the pair (R, P) will be called an ordered ring of level n.

Definition 1. Let both (R, P) and (S, Q) be ordered rings. (S, Q) is called an ordered extension of (R, P), if $R \subseteq S$, $Q \cap R = P$, and P and Q have the same exact level. In this case, we also say that Q is an extension of P to S.

Now we give the definition of real closed rings of higher level as follows:

Definition 2. An ordered ring (R, P) is called real closed, if there is no proper tight integral extension of R which possesses an ordering extending P.

Clearly, for an ordered field (F, P) of level n, (F, P) is real closed in the sense of Definition 2 if and only if (F, P) is a (generalized) real closed field in the usual sense.

For the existence of real closed rings of higher level, we obtain the following proposition.

Proposition 1.1. *Every ordered ring (R, P) of level n admits a real closed ordered extension $(\widehat{R}, \widehat{P})$ such that $\widehat{R}$ is a tight integral extension of R.*

Proof. Denote by $I(R)$ the totally integral closure of R. An easy application of Zorn's lemma shows that there exists a maximal subring $\widehat{R}$ of $I(R)$ containing R such that P can be extended to an ordering $\widehat{P}$ of $\widehat{R}$. By Lemma 3 in [17], $\widehat{R}$ is a tight integral extension of R.

Let (S, Q) be an ordered extension of $(\widehat{R}, \widehat{P})$ such that S is a tight integral extension of $\widehat{R}$. Then there is a ring homomorphism π of S into $I(R)$ such that $\pi(a) = a$ for all $a \in \widehat{R}$, since $I(R)$ is totally integrally closed. Suppose $\ker(\pi) \neq \{0\}$, where $\ker(\pi)$ is the kernel of π. Since S is a tight extension of $\widehat{R}$, we have $\ker(\pi) \cap \widehat{R} \neq \{0\}$, a contradiction. Thereby, π is an injection. It is easy to see that $(\pi(S), \pi(Q))$ is an ordered extension of $(\widehat{R}, \widehat{P})$. By the maximality of $(\widehat{R}, \widehat{P})$, we have $(\pi(S), \pi(Q)) = (\widehat{R}, \widehat{P})$ and $S = \widehat{R}$. Thus, there is no proper tight integral extension of $\widehat{R}$ which possesses an ordering extending $\widehat{P}$. By Definition 2, $(\widehat{R}, \widehat{P})$ is real closed. This completes the proof. □

In the sequel, for an ordered ring (R, P), such a real closed ring $(\widehat{R}, \widehat{P})$ as in Proposition 1.1 is called a real closure of (R, P).

Theorem 1.1. *Let (R, P) be an ordered ring, and let $I(R)$ be the totally integral closure of R. Then (R, P) is real closed if and only if for every prime ideal $\mathfrak{p}$ of R contained in $P \cap -P$, $\mathfrak{p}$ is also a prime ideal of $I(R)$ and $(R/\mathfrak{p}, P/\mathfrak{p})$ is real closed.*

Proof. *Necessity.* Assume that (R, P) is real closed. For every prime ideal $\mathfrak{p}$ of R contained in $P \cap -P$, there is a prime ideal $\mathfrak{q}$ of $I(R)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$, since $I(R)$ is integral over R. In this case, we have $(R + \mathfrak{q})/\mathfrak{q} \cong R/(\mathfrak{q} \cap R) = R/\mathfrak{p}$. Denote by τ the composition of the canonical homomorphisms: $R + \mathfrak{q} \to (R + \mathfrak{q})/\mathfrak{q} \to R/\mathfrak{p}$, and write

$\tau^{-1}(P/\mathfrak{p})$ for the preimage of $P/\mathfrak{p}$ under τ. Obviously, $\tau^{-1}(P/\mathfrak{p}) = P + \mathfrak{q}$. It is a routine to prove that $(R + \mathfrak{q}, P + \mathfrak{q})$ is an ordered extension of $(R, \mathfrak{p})$. By Lemma 3 in [17], $R + \mathfrak{q}$ is a tight integral extension of R. By Definition 2, we have $(R + \mathfrak{q}, P + \mathfrak{q}) = (R, P)$ and $R + \mathfrak{q} = R$. It follows that $\mathfrak{q} \subseteq R$. Thereby, $\mathfrak{p} = \mathfrak{q} \cap R = \mathfrak{q}$, i.e., $\mathfrak{p}$ is a prime ideal of $I(R)$.

According to Proposition 3 in [10] and Proposition 5 in [11], $I(R)/\mathfrak{p}$ is a totally integrally closed ring. From Lemma 1 in [17], it follows that $I(R)/\mathfrak{p}$ is a tight extension of $R/\mathfrak{p}$. This implies that $I(R)/\mathfrak{p}$ is the totally integral closure of $R/\mathfrak{p}$. Obviously, $P/\mathfrak{p}$ is an ordering of $R/\mathfrak{p}$, and $P/\mathfrak{p}$ has the same exact level as P. By Proposition 1.1, there is a real closure $(\overline{S}, \overline{Q})$ of $(R/\mathfrak{p}, P/\mathfrak{p})$ such that $\overline{S} \subseteq I(R)/\mathfrak{p}$. Clearly, $\overline{S}$ may be expressed as $\overline{S} = S/\mathfrak{p}$, where S is a subring of $I(R)$ containing R. Denote by Q the preimage of $\overline{Q}$ under the canonical homomorphism: $S \to S/\mathfrak{p}$. It is easy to see that (S, Q) is an ordered extension of (R, P). So we have $S = R$ and $Q = P$. Therefore, $(R/\mathfrak{p}, P/\mathfrak{p})$ $(= (\overline{S}, \overline{Q}))$ is real closed.

Sufficiency. Assume that the conditions stated in Theorem 1.1 are satisfied by every prime ideal of R contained in $P \cap -P$. By Proposition 1.1, there is a real closure (S, Q) of (R, P) such that $S \subseteq I(R)$. Put $\mathfrak{q} := Q \cap -Q$ and $\mathfrak{p} = \mathfrak{q} \cap R$. Clearly, $\mathfrak{p}$ is a real prime ideal of R. Moreover, we have $\mathfrak{p} = (Q \cap -Q) \cap R = P \cap -P$. By the assumption, $\mathfrak{p}$ is a prime ideal of $I(R)$ and $(R/\mathfrak{p}, P/\mathfrak{p})$ is real closed. Since $I(R)$ is integral over S, there exists a prime ideal $\mathfrak{q}'$ of $I(R)$ such that $\mathfrak{q}' \cap S = \mathfrak{q}$. It follows that $\mathfrak{q}' \cap R = (\mathfrak{q}' \cap S) \cap R = \mathfrak{q} \cap R = \mathfrak{p} = \mathfrak{p} \cap R$. By Corollary 5.9 in [1], we have $\mathfrak{q}' = \mathfrak{p}$. This yields $\mathfrak{q} = \mathfrak{p}$. In this case, $I(R)/\mathfrak{p}$ is the totally integral closure of $R/\mathfrak{p}$, and $S/\mathfrak{p}$ is a tight integral extension of $R/\mathfrak{p}$. It is easy to see that $(S/\mathfrak{p}, Q/\mathfrak{p})$ is an ordered extension of $(R/\mathfrak{p}, P/\mathfrak{p})$. Since $(R/\mathfrak{p}, P/\mathfrak{p})$ is real closed, we have $S/\mathfrak{p} = R/\mathfrak{p}$ and $Q/\mathfrak{p} = P/\mathfrak{p}$. It follows that $(S, Q) = (R, P)$. This implies that (R, P) is real closed. The proof is completed. □

Actually, Theorem 1.1 may further be improved as follows:

Theorem 1.2. *Let (R, P) be an ordered ring, and let $I(R)$ be the totally integral closure of R. Then the following statements are equivalent*:

(1) *(R, P) is real closed*;
(2) *For every prime ideal $\mathfrak{p}$ of R contained in $P \cap -P$, $\mathfrak{p}$ is a prime ideal of $I(R)$ and $(R/\mathfrak{p}, P/\mathfrak{p})$ is real closed*;
(3) *For some prime ideal $\mathfrak{p}$ of R contained in $P \cap -P$, $\mathfrak{p}$ is a prime ideal of $I(R)$ and $(R/\mathfrak{p}, P/\mathfrak{p})$ is real closed.*

Proof.

(1) $\Rightarrow$ (2) follows immediately from Theorem 1.1.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). Assume that $\mathfrak{p}$ is a prime ideal of $I(R)$ and $(R/\mathfrak{p}, P/\mathfrak{p})$ is real closed for some prime ideal $\mathfrak{p}$ of R contained in $P \cap -P$. By Proposition 1.1, there is a real closure (S, Q) of (R, P) such that $S \subseteq I(R)$. Since $\mathfrak{p}$ is a prime ideal of $I(R)$, $I(R)/\mathfrak{p}$ is the totally integral closure of $R/\mathfrak{p}$, and $S/\mathfrak{p}$ is a tight integral extension of $R/\mathfrak{p}$. Observe that $(S/\mathfrak{p}, Q/\mathfrak{p})$ is an ordered extension of $(R/\mathfrak{p}, P/\mathfrak{p})$. By the assumption that $(R/\mathfrak{p}, P/\mathfrak{p})$ is

real closed, we have $S/\mathfrak{p} = R/\mathfrak{p}$ and $Q/\mathfrak{p} = P/\mathfrak{p}$. This implies $(S, Q) = (R, P)$; hence (R, P) is real closed. The proof is completed. $\square$

Now let (R, P) be an ordered ring with $P \cap -P = \{0\}$. Since $P \cap -P$ is a prime ideal, R is a integral domain. Denote by F the fraction field of R. Then, F possesses the following ordering induced by P:

$$P_F = \left\{ \frac{a}{b} \,\middle|\, a, b \in P \text{ and } b \neq 0 \right\}.$$

Obviously, P and P_F have the same exact level. For the sake of convenience, the ordered field (F, P_F) will be called the ordered fraction field of (R, P) in the sequel.

Proposition 1.2. *Let (R, P) be an ordered ring with $P \cap -P = \{0\}$, and (F, P_F) the ordered fraction field of (R, P). Then (R, P) is real closed if and only if (F, P_F) is real closed, and R is integrally closed in F.*

Proof. Denote by Ω the algebraic closure of F. Then, Ω is the totally integral closure of F. Write $I(R)$ for the integral closure of R in Ω. Clearly, $I(R)$ is the totally integral closure of R.

Assume that (R, P) is real closed. By Proposition 1.1 and its proof, (F, P_F) admits a real closure $(\widehat{F}, \widehat{P_F})$ such that $\widehat{F} \subseteq \Omega$. Write S for the integral closure of R in $\widehat{F}$, and put $Q = \widehat{P_F} \cap S$. It is easy to verify that (S, Q) is an ordered extension of (R, P). Observe that S is a tight integral extension of R. Since (R, P) is real closed, we have $(R, P) = (S, Q)$ and $R = S$. This implies $(\widehat{F}, \widehat{P_F}) = (F, P_F)$, because $\widehat{F}$ is also the fraction field of R. Therefore, (F, P_F) is real closed, and R is integrally closed in F.

Conversely, assume that (F, P_F) is real closed and that R is integrally closed in F. By Proposition 1.1, there is a real closure (S, Q) of (R, P) such that $S \subseteq I(R)$. Suppose $Q \cap -Q \neq \{0\}$. Since S is a tight extension of R, we have $(Q \cap -Q) \cap R \neq \{0\}$, i.e., $P \cap -P \neq \{0\}$. This contradicts the hypothesis. Hence $Q \cap -Q = \{0\}$. Let (K, Q_K) be the ordered fraction field of (S, Q). Then, (K, Q_K) is obviously an ordered extension of (F, P_F). This yields $(K, P_K) = (F, P_F)$, since (F, P_F) is real closed. This implies $S \subseteq F$. Since R is integrally closed in F and S is integral over R, we have $S \subseteq R$ and $S = R$. Therefore, (R, P) $(= (S, Q))$ is real closed. This completes the proof. $\square$

Corollary. *Let (R, P) be a real closed ring of level n with $P \cap -P = \{0\}$, and S a multiplicatively closed subset in $\dot{R}$. Then (R_S, P_S) is also a real closed ring of level n with $P_S \cap -P_S = \{0\}$, where R_S is the fraction ring of R with respect to S, and $P_S := \{a/s^{2n} \mid a \in P$ and $s \in S\}$.*

Proof. Obviously, (R_S, P_S) is an ordered ring of level n with $P_S \cap -P_S = \{0\}$. Denote by (F, P_F) the ordered fraction field of (R, P). It is easy to see that (F, P_F) is also the ordered fraction field of (R_S, P_S). By Proposition 1.2, (F, P_F) is real closed, and R is integrally closed in F. Observe that R_S is also integrally closed in F. Again by Proposition 1.2, (R_S, P_S) is also a real closed ring of level n. This completes the proof. $\square$

Proposition 1.3. *If (R, P) is a real closed ring, then $P \cap -P$ is the only real ideal contained in $P \cap -P$, and $P \cap -P = R^{2m} \cap -R^{2m}$ for every $m \in \mathbb{N}$.*

Proof. Denote by $I(R)$ the totally integral closure of R. For $a \in P \cap -P$, by Theorem 1 and Corollary 1 in [11], the monic polynomial $x^{2m} - a$ has a root α in $I(R)$ for every $m \in \mathbb{N}$. In this case, $\alpha^{2m} = a \in P \cap -P$. By Theorem 1.1, $P \cap -P$ is also a prime ideal of $I(R)$. So we have $\alpha \in P \cap -P \subseteq R$. Thus $a = \alpha^{2m} \in R^{2m}$. Likewise, $-a \in R^{2m}$. This implies $a \in R^{2m} \cap -R^{2m}$, and $P \cap -P \subseteq R^{2m} \cap -R^{2m}$. For $a \in R^{2m} \cap -R^{2m}$, we have $a = b^{2m} = -c^{2m}$ where $b, c \in R$. It follows that $b^{2m} + c^{2m} = 0 \in P \cap -P$. Since $P \cap -P$ is a real prime ideal of R, we have $b^m \in P \cap -P$, and $a = (b^m)^2 \in P \cap -P$. Thereby, $P \cap -P = R^{2m} \cap -R^{2m}$ for every $m \in \mathbb{N}$.

Let $\mathfrak{p}$ be any real ideal of R such that $\mathfrak{p} \subseteq P \cap -P$. By the reality of $\mathfrak{p}$, it is easy to check $R^2 \cap -R^2 \subseteq \mathfrak{p}$. By the preceding argument, we have $P \cap -P = R^2 \cap -R^2 \subseteq \mathfrak{p} \subseteq P \cap -P$. Therefore $\mathfrak{p} = P \cap -P$. $\square$

2. Real valuations on real closed rings of higher level

In this section, we shall investigate real closed rings of higher level via real valuations. The discussion in this section is closely related to the theory of real closed fields of higher level. Many important results on real closed fields of higher level may be found in the relevant literature, e.g., [2–7].

In the sequel, all valuations on a commutative ring mean Manis valuations in the sense of Manis [13]. Details on valuations of commutative rings can be found in [13]. Let Γ be an ordered abelian group, written additively, and let $\Gamma_\infty = \Gamma \cup \{\infty\}$, where $\gamma + \infty = \infty + \gamma = \infty$ and $\gamma < \infty$ for all $\gamma \in \Gamma$. A map v of a ring R onto Γ_∞ is called a (Manis) valuation on R, if $v(ab) = v(a) + v(b)$ and $v(a + b) \geqslant \min\{v(a), v(b)\}$ for all $a, b \in R$. In this case, Γ is called the value group of v. For a valuation v on R, we have the three subsets of R as follows: $A_v := \{a \in R \mid v(a) \geqslant 0\}$, $M_v := \{a \in R \mid v(a) > 0\}$, and $C_v := \{a \in R \mid v(a) = \infty\}$. It is easy to check that A_v is a subring of R, M_v is a prime ideal of A_v, C_v is a prime ideal of both R and A_v, and $C_v \subseteq M_v \subset A_v$. According to [13], such a pair (A_v, M_v) is called a valuation pair of R (precisely, the valuation pair of v), and C_v is called the core of v. Let v be a valuation on R. According to Refs. [8,14], v is called real if A_v/M_v is a real domain, where (A_v, M_v) is the valuation pair of v. It is clear that a valuation v on R is real if and only if $v(a^2 + b^2) = 2\min\{v(a), v(b)\}$ for all $a, b \in R$.

According to Definition 3 in [19], the compatibility between valuations and orderings of a commutative ring may be defined as follows:

Definition 3. For an ordered ring (R, P) of level n and a valuation $v: R \to \Gamma_\infty$, we say that v is compatible with P, if $v(p_1 + p_2) = \min\{v(p_1), v(p_2)\}$ for all $p_1, p_2 \in P$.

By Proposition 3 in [19], we have the fact as follows: Let (R, P) be an ordered ring, and v a valuation on R with valuation pair (A_v, M_v). Then v is compatible with P if and only if $P \cap -P \subseteq v^{-1}(\infty)$ and $P \cap (A_v \setminus M_v) + M_v \subseteq P$. In particular, when R is a field,

v is compatible with P if and only if $1 + M_v \subseteq P$. Therefore, Definition 3 is coincident with that of the compatibility in [3] in the category of fields.

Now let (R, P) be an ordered ring of level n, and let v be a valuation on R with valuation pair (A, M). Then we may construct the subset P_v of A/M as follows:

$$P_v = \{\bar{a} \in A/M \mid a \in A \cap P\},$$

where $\bar{a} := a + M$ for every $a \in A$.

Let C denote the core of v. Then, v induces a valuation w on the domain R/C by defining $w(a + C) = v(a)$ for all $a \in R$. Denote by F the fraction field of R/C. Clearly, a valuation v^* on F may be induced in the following way:

$$\text{for } a \in R \text{ and } b \in R \setminus C, \quad v^*\left(\frac{a+C}{b+C}\right) = w(a+C) - w(b+C).$$

Obviously, the value groups of both w and v^* are the same as v, and $(A^*_{T^*}, M^*_{T^*})$ is the valuation pair of v^*, where $A^* := A/C$, $M^* := M/C$ and T^* is the image of $A \setminus M$ under the canonical homomorphism $R \to R/C$. Write F_{v^*} for the residue field $A^*_{T^*}/M^*_{T^*}$. By the permutability of residue class ring and fraction ring formation, we have $A^*_{T^*}/M^*_{T^*} \cong (A^*/M^*)_{\widehat{T}}$, where $\widehat{T}$ is the image of T^* under the canonical homomorphism $A^* \to A^*/M^*$. Further, we have $(A^*/M^*)_{\widehat{T}} \cong (A/M)_{\overline{T}}$, where $\overline{T}$ is the image of $\widehat{T}$ under the canonical isomorphism $A^*/M^* \to A/M$. Hence $F_{v^*} \cong (A/M)_{\overline{T}}$. Observe that $(A/M)_{\overline{T}}$ is just the fraction field of A/M. Thereby, F_{v^*} may be regarded as the fraction field of A/M.

By a routine verification, we may establish the following result.

Lemma 1. *Let the notations be as above, let F be the fraction field of R/C, and write P_F for the subset $\{(a + C)/(b + C) \mid a, b \in P \text{ and } b \notin C\}$ of F. Then the following statements are equivalent*:

(1) *v is compatible with P*;
(2) *P_v is an ordering of A/M such that $P_v \cap -P_v = \{0\}$*;
(3) *P_F is an ordering of F, and v^* is compatible with P_F.*

Theorem 2.1. *Let (R, P) be a real closed ring of level n. Then, every real valuation v on R is compatible with P.*

Proof. Put $\wp = P \cap -P$, and denote by (F, P_F) the ordered fraction field of $(R/\wp, P/\wp)$. By Theorem 1.1 and Proposition 1.2, (F, P_F) is a real closed field, and $R/\wp$ is integrally closed in F. Denote by u the canonical valuation induced by P_F. Then $(A(P_F), M(P_F))$ is the valuation pair of u, and F_u is the residue field of u where $F_u = A(P_F)/M(P_F)$.

First, we may assert that $P \cap -P \subseteq v^{-1}(\infty)$. Indeed, if $z \in P \cap -P$, by Proposition 1.3 we have $z = a^{2n} = -b^{2n}$ for some $a, b \in R$. Since v is a real valuation, $v(z) = \min\{v(a^{2n}), v(b^{2n})\} = v(a^{2n} + b^{2n}) = v(0) = \infty$, and $z \in v^{-1}(\infty)$. Hence $P \cap -P \subseteq v^{-1}(\infty)$.

Let $a, b \in P$. If either $a \in v^{-1}(\infty)$ or $b \in v^{-1}(\infty)$, necessarily $v(a+b) = \min\{v(a), v(b)\}$. Now assume $a \notin v^{-1}(\infty)$ and $b \notin v^{-1}(\infty)$. By the preceding argument, we have $a \notin \wp$ and $b \notin \wp$. For any $y \in R$, write $\bar{y}$ for the element $y + \wp$ in $R/\wp$. Obviously, $\bar{a}, \bar{b} \in P_F$. By the compatibility of u with P_F, we have $u(\bar{a} + \bar{b}) = \min\{u(\bar{a}), u(\bar{b})\}$. Without loss of generality, we may assume $u(\bar{a}) \leqslant u(\bar{b})$, i.e., $u(\bar{a} + \bar{b}) = u(\bar{a})$. Put $z := (\bar{a} + \bar{b})/\bar{a}$. Then $z \in A_u \setminus M_u$. Thereby $\hat{z} := z + M(P_F)$ is a non-zero element of F_u. By Proposition 2.4 and Satz 2.2 in [3], F_u^2 is an archimedean ordering of F_u. Thereby, there exists an $m \in \mathbb{N}$ such that $m\hat{z}^2 - 1 \in \dot{F}_u^2$. By Theorem 3.8 in [6], u is a henselian valuation. With the aid of Hensel's lemma, we have $mz^2 - 1 \in \dot{F}^2$. This yields $m(\bar{a} + \bar{b})^2 - \bar{a}^2 \in \dot{F}^2$. Since $R/\wp$ is integrally closed in F, we further have $m(\bar{a} + \bar{b})^2 - \bar{a}^2 \in (R/\wp)^2$, i.e., $m(\bar{a} + \bar{b})^2 - \bar{a}^2 = \bar{c}^2$ for some $c \in R$. Write $I(R)$ for the totally integral closure of R. By Theorem 1 and Corollary 1 in [11], the monic polynomial $x^2 - m(a+b)^2 + a^2$ has a root α in $I(R)$. By Theorem 1.1, $\wp$ is also a prime ideal of $I(R)$. It follows that $\alpha^2 + \wp = m(a+b)^2 - a^2 + \wp = c^2 + \wp$, and $\alpha + \wp = \pm c + \wp$. Hence $\alpha \in R$. By the reality of v, we have $2v(a+b) = v(m(a+b)^2) = v(\alpha^2 + a^2) = \min\{v(\alpha^2), v(a^2)\} \leqslant v(a^2) = 2v(a)$, and $v(a+b) \leqslant v(a)$. Likewise, we have $v(a) \leqslant v(a+b)$, since we also have $\bar{a}/(\bar{a} + \bar{b}) \in A_u \setminus M_u$. Thereby, $v(a+b) = v(a)$. Moreover, we have $v(b) = v((a+b) - a) \geqslant \min\{v(a+b), v(a)\} = v(a)$. So we have $v(a+b) = v(a) = \min\{v(a), v(b)\}$. Therefore, v is compatible with P. This completes the proof. $\square$

In order to characterize real closed rings via real valuations, we need the following.

Definition 4. Let v be a valuation on a ring R with valuation pair (A_v, M_v), and denote by λ the canonical homomorphism of $A_v[x]$ onto $A_v/M_v[x]$. v is called henselian, if the following condition is satisfied: For any monic polynomials $f(x) \in A_v[x]$ and $\phi(x)$, $\psi(x) \in A_v/M_v[x]$ such that $\lambda(f(x)) = \phi(x)\psi(x)$ and $\phi(x)$, $\psi(x)$ are relatively prime, there exist monic polynomials $g(x), h(x) \in A_v[x]$ such that $f(x) = g(x)h(x)$, $\lambda(g(x)) = \phi(x)$ and $\lambda(h(x)) = \psi(x)$.

In Ref. [19], the following lemma was established.

Lemma 2. *If v is a henselian valuation on R with core C_v, then v^* is a henselian valuation on F, where F is the fraction field of R/C_v, and v^* is the valuation on F induced by v as above.*

For the further investigations, some results on real closed fields of higher level are needed.

Let (F, T) be a preordered field of level n, let v be a valuation on F with valuation ring A_v and valuation ideal M_v, and G_v the (additive) value group of v. Denote $F_v = A_v/M_v$, and for every $a \in A_v$, write $\bar{a}$ for the element $a + M_v$ in A_v/M_v. Then we may obtain the following subset of A_v/M_v:

$$T_v = \{\bar{a} \mid a \in T \cap A_v\}.$$

Obviously, $F_v^{2n} \subseteq T_v$, $T_v + T_v \subseteq T_v$, and $T_v \cdot T_v \subseteq T_v$. In Ref. [3], the following result is established:

Proposition 2.1. *Let the notations be as above. If $1 + M_v \subseteq T$, then we have*

(1) *T_v is a preordering of level n of F_v.*
(2) *There is such an exact sequence of groups as follows*:

$$1 \longrightarrow \dot{F}_v/\dot{T}_v \xrightarrow{\tau} \dot{F}/\dot{T} \xrightarrow{\bar{v}} G_v/v(\dot{T}) \longrightarrow 1,$$

where $\tau(\bar{a}\dot{T}_v) = a\dot{T}$ for every $\bar{a} \in \dot{F}_v$, and $\bar{v}(a\dot{T}) = v(a) + v(\dot{T})$ for every $a \in \dot{F}$.

According to Satz 2.2 in [3], for an ordered field (F, P), the following subset of F is a valuation ring compatible with P:

$$A(P) = \{x \in F \mid n \pm x \in P \text{ for some } n \in \mathbb{N}\}.$$

Denote by u the associated valuation of $A(P)$. Then u is a valuation on F compatible with P, and its valuation ideal is as follows:

$$M(P) = \{x \in F \mid 1 \pm nx \in P \text{ for all } n \in \mathbb{N}\}.$$

In the sequel, the valuation u on F is called the canonical valuation induced by P.

Proposition 2.2. *Let (F, P) be an ordered field of exact level n, denote by u the canonical valuation induced by P, and write F_u and Γ for the residue field and value group of u respectively. Then (F, P) is real closed if and only if the following statements are true*:

(1) *u is a henselian valuation on F.*
(2) *F_u is an ordinary real closed field.*
(3) *For every prime p, $\Gamma = p\Gamma$ if $p \nmid n$ or $|\Gamma/p\Gamma| = p$ if $p \mid n$.*

Proof. The conclusion follows from Theorem 3.8 in [6] and its proof. □

Actually, Proposition 2.2 may be modified as follows:

Proposition 2.3. *Let (F, P) be an ordered field of exact level n, let v be a real valuation on F, and denote by A_v, M_v and G_v the valuation ring, ideal and value group of v respectively. Then (F, P) is real closed if and only if the following conditions are satisfied*:

(1) *For some divisor m of n, (F_v, P_v) is a real closed field of exact level m, where $F_v = A_v/M_v$, and $P_v = \{a + M_v \mid a \in P \cap A_v\}$;*
(2) *v is a henselian valuation*;
(3) *For every prime p, $[G_v : pG_v] = p$ if $p \mid n$ but $p \nmid m$; otherwise $pG_v = G_v$.*

Proof. Denote by u the canonical valuation induced by P, and write Γ for the value group of u. Then, $A(P)$ and $M(P)$ are the valuation ring and ideal of u respectively. It is well known that $A(P) \subseteq A_v$. Put $H := u(A_v \setminus M_v)$. In valuation theory, the following facts are familiar: (1) M_v is a prime ideal of $A(P)$ such that $M_v \subseteq M(P)$; (2) $A_v = A(P)_{M_v}$ the localization of $A(P)$ at M_v; (3) u induces a valuation $\bar{u}$ on A_v/M_v with valuation ring $A(P)/M_v$, valuation ideal $M(P)/M_v$ and value group H such that $\bar{u}(a + M_v) = u(a)$ for every $a \in A_v \setminus M_v$; (4) H is a convex subgroup of Γ, and there is a group isomorphism of G_v to Γ/H such that $v(a) \mapsto u(a) + H$ for every $a \in \dot{F}$. Moreover, it is easy to see that $\bar{u}$ is just the canonical valuation induced by P_v.

Necessity. Assume that (F, P) is real closed. By Theorem 2.1, v is compatible with P. Then we have $1 + M_v \subseteq P$. According to Proposition 2.1(2), $\dot{F}_v/\dot{P}_v$ is isomorphic to a subgroup of $\dot{F}/\dot{P}$. Since $\dot{F}/\dot{P}$ is a cyclic group of order $2n$, $\dot{F}_v/\dot{P}_v$ is also a cyclic group. Since $-\dot{P}_v$ is an element of order 2 in $\dot{F}_v/\dot{P}_v$, the order of $\dot{F}_v/\dot{P}_v$ is even. Put $|\dot{F}_v/\dot{P}_v| = 2m$. Then m is a divisor of n, and (F_v, P_v) is an ordered field of exact level m.

Let (K, w) be a henselization of (F, v). Then (K, w) is an immediate extension of (F, v), i.e., G_w and K_w $(= A_w/M_w)$ may be identified with G_v and F_v $(= A_v/M_v)$ respectively. By fact (3) as above, u induces a valuation $\bar{u}$ on A_v/M_v with valuation ring $A(P)/M_v$, valuation ideal $M(P)/M_v$ and value group H such that $\bar{u}(a + M_v) = u(a)$ for every $a \in A_v \setminus M_v$. By the isomorphism $A_w/M_w = (A_v + M_w)/M_w \cong A_v/M_v$, $(A(P) + M_w)/M_w$ is a valuation ring of A_w/M_w. Then it is easy to see that $A(P) + M_w$ is a valuation ring of K such that $(A(P) + M_w) \cap F = A(P)$. Denote by $\hat{u}$ the associated valuation of $A(P) + M_w$, and write $K_{\hat{u}}$ for the residue field of $\hat{u}$. Then $\hat{u}$ is an extension of u to K. Obviously, $M_{\hat{u}} = M(P) + M_w$, i.e., $M(P) + M_w$ is the maximal ideal of $A(P) + M_w$, and $K_{\hat{u}} = A_{\hat{u}}/M_{\hat{u}} = (A(P) + M_w)/(M(P) + M_w) \cong A(P)/M(P) = F_u$. By Proposition 2.2, u is a henselian valuation on F, and F_u is a real closed field. Hence $\hat{u}$ is a henselian valuation on K, and $K_{\hat{u}}$ is real closed. By identifying $K_{\hat{u}}$ with F_u, we have $P_u = K_{\hat{u}}^2 = K_{\hat{u}}^{2n}$.

Put $Q = PK^{2n}$, i.e., $Q = \{p\alpha^{2n} \mid p \in P \text{ and } \alpha \in K\}$. Obviously, $K^{2n} \subseteq Q$, and $Q \cdot Q \subseteq Q$. Let $p \in \dot{P}$, and $\alpha \in \dot{K}$. Without loss of generality, we may assume $\hat{u}(p\alpha^{2n}) \geqslant 0$. When $\hat{u}(p\alpha^{2n}) > 0$,

$$\left(1 + p\alpha^{2n}\right) + M_{\hat{u}} = 1 + M_{\hat{u}} \in \dot{K}_{\hat{u}}^{2n}.$$

By Hensel's lemma, we have $1 + p\alpha^{2n} \in A_{\hat{u}}^{2n} \subseteq Q$. Now assume that $\hat{u}(p\alpha^{2n}) = 0$. In this case, $w(p\alpha^{2n}) \geqslant 0$. Since $G_w = G_v$, there is an $a \in \dot{F}$ such that $w(\alpha) = w(a)$. Then

$$\left(1 + p\alpha^{2n}\right) + M_w = \left(1 + \left(pa^{2n}\right)\left(\alpha a^{-1}\right)^{2n}\right) + M_w \in \dot{P}_v,$$

since $(pa^{2n}) + M_w \equiv (pa^{2n}) + M_v \in P_v$ and $(\alpha a^{-1})^{2n} + M_w \in K_w^{2n} \equiv F_v^{2n} \subseteq P_v$. Hence $1 + p\alpha^{2n} = p_1 + \beta$ for some $p_1 \in P \cap A_w$ and some $\beta \in M_w$. Since $\beta \in M_w \subseteq M_{\hat{u}}$, we have

$$\hat{u}(p_1) = \hat{u}\left(1 + p\alpha^{2n} - \beta\right) = \hat{u}\left(1 + p\alpha^{2n}\right) = 0,$$

and $1 + p\alpha^{2n} + M_{\hat{u}} = p_1 + M_{\hat{u}} \equiv p_1 + M_u \in \dot{P}_u = \dot{K}_{\hat{u}}^{2n}$. From Hensel's lemma, it follows that $1 + p\alpha^{2n} \in A_{\hat{u}}^{2n} \subseteq Q$. So we always have $1 + Q \subseteq Q$. This implies $Q + Q \subseteq Q$.

Likewise, it is easy to see that $-1 \notin Q$, and $1 + M_w \subseteq Q$. Then Q is a preordering of level n of K. Moreover, it is easy to verify that $Q \cap F = P$, $Q_w = P_v$, and $w(\dot{Q}) = v(\dot{P})$, since $G_w = G_v$. According to Proposition 2.1(2), we have

$$|\dot{K}/\dot{Q}| = |\dot{K}_w/\dot{Q}_w| \cdot \left|G_w/w(\dot{Q})\right| = |\dot{F}_v/\dot{P}_v| \cdot \left|G_v/v(\dot{P})\right| = |\dot{F}/\dot{P}|.$$

This implies that (K, Q) is an ordered extension of (F, P). Since (F, P) is real closed, we have $K = F$. This shows that v is a henselian valuation.

Suppose that (F_v, P_v) is not a real closed field. Then, there is a finite extension L of F_v such that $[L : F_v] > 1$, and P_v may be extended to an ordering Q_0 of L. By a familiar fact in valuation theory, there is a finite extension K of F such that $[K : F] = [L : F_v]$, and v can be extended to a valuation w on K with residue field L. By Corollary 20.23 in [9], $[K : F] = [L : F_v] \cdot [G_w : G_v]$, where G_w is the value group of w. This yields $G_w = G_v$.

Denote by A_w and M_w the valuation ring and ideal of w respectively, and put $A_{\tilde{u}} := \{\alpha \in A_w \mid (n \pm \alpha) + M_w \in Q_0 \text{ for some } n \in \mathbb{N}\}$. It is easy to prove that $A_{\tilde{u}}$ is a valuation ring of K contained in A_w, and its residue field, denoted by $K_{\tilde{u}}$, is isomorphic to that of the canonical valuation of L induced by Q_0. Thereby, $A_{\tilde{u}}$ is a real valuation ring. Moreover, it is easy to verify that $A_{\tilde{u}} \cap F = A(P)$. Hence, $K_{\tilde{u}}$ is a real algebraic extension of F_u, and $K_{\tilde{u}} = F_u$, since F_u is real closed. Write $\tilde{u}$ for the associated valuation of $A_{\tilde{u}}$. Then $\tilde{u}$ is henselian, since u is henselian.

Put $\Delta = \{\alpha \in A_w \mid \alpha \notin M_w,\ \text{but}\ \alpha + M_w \in Q_0\}$, and $Q := P\Delta K^{2n}$ $(= \{p\delta\alpha^{2n} \mid p \in P,\ \delta \in \Delta,\ \text{and}\ \alpha \in K\})$. Obviously, $K^{2n} \subseteq Q$, and $Q \cdot Q \subseteq Q$. By the argument similar to the preceding paragraph, it may be proved that $Q + Q \subseteq Q$, $-1 \notin Q$, $1 + M_w \subseteq Q$, and $Q \cap F = P$. Then Q is a preordering of level n of K. It is a routine to verify that

$$\dot{Q}_w = Q_0, \quad \text{and} \quad w(\dot{Q}) = w(\dot{P})w(\Delta)w\big(\dot{K}^{2n}\big) = v(\dot{P})(2nG_w) = v(\dot{P})(2nG_v) = v(\dot{P}).$$

According to Proposition 2.1(2), we have

$$\begin{aligned} |\dot{K}/\dot{Q}| &= |\dot{K}_w/\dot{Q}_w| \cdot \left|G_w/w(\dot{Q})\right| = |\dot{K}_w/\dot{Q}_0| \cdot \left|G_v/v(\dot{P})\right| = |\dot{F}_v/\dot{P}_v| \cdot \left|G_v/v(\dot{P})\right| \\ &= |\dot{F}/\dot{P}|. \end{aligned}$$

This implies that (K, Q) is a proper ordered extension of (F, P), a contradiction. Thus, (F_v, P_v) is a real closed field.

Let p be a prime. If $p \nmid n$, by Proposition 2.2 we have $\Gamma = p\Gamma$. Then $\Gamma/H = p(\Gamma/H)$ where $H = u(A_v \setminus M_v)$. By the isomorphism $G_v \cong \Gamma/H$, we have $G_v = pG_v$. Now assume $p \mid n$. By Proposition 2.2, $|\Gamma/p\Gamma| = p$. Let $u(a) + p\Gamma$ be a generator of the cyclic group $\Gamma/p\Gamma$, where $a \in \dot{F}$. Then, for every $z \in \dot{F}$, $u(z) = ru(a) + pu(b)$ for some $b \in \dot{F}$ and some $r \in \mathbb{N}$ with $0 \leqslant r < p$. Thereby, $za^{-r}b^{-p}$ is invertible in $A(P)$, hence in A_v. This yields $v(za^{-r}b^{-p}) = 0$, and $v(z) = rv(a) + pv(b) \in rv(a) + pG_v$. It follows that $|G_v/pG_v| = 1$ or p. Further consider the two possible subcases as follows:

Subcase 1: $p \nmid m$. Suppose $|G_v/pG_v| = 1$. Then, for any $z \in \dot{F}$, there is an $b \in \dot{F}$ such that $zb^{-p} \in A_v/M_v$. By the proof of Satz 3.6 in [3], we have either $F_v = F_v^p \cup -F_v^p$ if $p = 2$ or $F_v = F_v^p$ if $p \neq 2$, since (F_v, P_v) is real closed. By Hensel's lemma, we have either $zb^{-p} \in A_v^p$ or $zb^{-p} \in -A_v^p$. This yields $u(z) \in p\Gamma$. By the arbitrariness of z, we have $\Gamma = p\Gamma$, a contradiction. Thus $|G_v/pG_v| = p$.

Subcase 2: $p \mid m$. By Proposition 2.2, $|H/pH| = p$. Let $h + pH$ be a generator of the cyclic group H/pH where $h \in H$, and r the order of $h + p\Gamma$. Obviously, $r \mid p$, and $r = 1$ or $r = p$. Suppose $r = 1$. Then $h \in p\Gamma$, and $h = p\gamma$ for some $\gamma \in \Gamma$. By the convexity of H in Γ, we have $\gamma \in H$, and $h \in pH$, a contradiction. This yields $r = p$. Since $\Gamma/p\Gamma$ is the cyclic group of order p, $h + p\Gamma$ is a generator of $\Gamma/p\Gamma$. Thereby, $\Gamma = p\Gamma + H$, and $\Gamma/H = p(\Gamma/H)$. This implies $G_v = pG_v$.

Sufficiency. Observe that $\bar{u}$ is the valuation induced by P_v and (F_v, P_v) is real closed. By Proposition 2.2, the residue field $(A(P)/M_v)/(M(P)/M_v)$ of $\bar{u}$ is a real closed field. By the isomorphism $A(P)/M(P) \cong (A(P)/M_v)/(M(P)/M_v)$, F_u is a real closed field.

Let $f(x) \in A(P)[x]$ be a monic polynomial, and $\bar{f}(x) = \phi(x)\psi(x)$, where $\bar{f}(x)$ is the image of $f(x)$ under the canonical homomorphism of $A(P)[x]$ to $A(P)/M(P)[x]$, and $\phi(x)$, $\psi(x) \in A(P)/M(P)[x]$ are two relatively prime monic polynomials. By Proposition 2.2, $\bar{u}$ is henselian. By the isomorphism $(A(P)/M_v)/(M(P)/M_v) \cong A(P)/M(P)$, we may regard $\bar{f}(x)$, $\phi(x)$ and $\psi(x)$ as polynomials in $(A(P)/M_v)/(M(P)/M_v)[x]$. By Hensel's lemma for $\bar{u}$, we have $\hat{f}(x) = \Phi(x)\Psi(x)$, where $\hat{f}(x)$ is the image of $f(x)$ under the canonical homomorphism of $A(P)[x]$ to $A(P)/M_v[x]$, and $\Phi(x)$, $\Psi(x) \in A(P)/M_v[x]$ such that $\phi(x)$, $\psi(x)$ are respectively the images of $\Phi(x)$, $\Psi(x)$ under the canonical homomorphism of $A(P)/M_v[x]$ to $A(P)/M(P)[x]$. Clearly, $\Phi(x)$ and $\Psi(x)$ are two relatively prime monic polynomials. By condition (2), we further have $f(x) = g(x)h(x)$, where $g(x)$, $h(x) \in A_v[x]$ such that $\Phi(x)$, $\Psi(x)$ are respectively the images of $g(x)$, $h(x)$ under the canonical homomorphism of $A_v[x]$ to $A_v/M_v[x]$. Since $A(P)$ is integrally closed in F, $g(x)$, $h(x) \in A(P)[x]$. Therefore, u is a henselian valuation.

For a prime p, we consider the three possible cases as follows.

Case 1: $p \nmid n$. For any $a \in \dot{F}$, by condition (3) we have $v(a) = pv(b)$ for some $b \in \dot{F}$. Then $ab^{-p} + M_v \in F_v$. Obviously, $p \nmid m$. According to Proposition 2.2, $\bar{u}(ab^{-p} + M_v) \in H = pH$, and $u(a) \in pu(b) + pH \subseteq p\Gamma$. In this case, we have $\Gamma = p\Gamma$.

Case 2: $p \mid n$ and $p \mid m$. For any $z \in \dot{F}$, by condition (3) we have $v(z) = pv(b)$ for some $b \in \dot{F}$. Then $zb^{-p} + M_v \in F_v$. According to Proposition 2.2, H/pH is a cyclic group of order p. Let $h + pH$ be a generator of H/pH, where $h \in H$. Then $\bar{u}(zb^{-p} + M_v) + pH = rh + pH$ for some integer r with $0 \leqslant r < p$. This yields $u(z) + p\Gamma = rh + p\Gamma$. By the arbitrariness of z, we have $|\Gamma/p\Gamma| = 1$ or p. Suppose $|\Gamma/p\Gamma| = 1$. Then $h \in p\Gamma$. By the convexity of H in Γ, we may deduce a contradiction: $h \in pH$. So we have $|\Gamma/p\Gamma| = p$.

Case 3: $p \mid n$ but $p \nmid m$. In this case, by condition (3), G_v/pG_v is a cyclic group of order p. Let $v(a) + pG_v$ be a generator of G_v/pG_v, where $a \in \dot{F}$. Then, for any $z \in \dot{F}$, $v(z) - rv(a) = pv(b)$ for some $b \in \dot{F}$ and some integer r with $0 \leqslant r < p$. Hence $za^{-r}b^{-p} \in A_v \setminus M_v$. According to Proposition 2.2, $\bar{u}(za^{-r}b^{-p} + M_v) \in H = pH$. This yields $u(z) + p\Gamma = ru(a) + p\Gamma$. By the arbitrariness of z, we have $|\Gamma/p\Gamma| = 1$ or p. Suppose $|\Gamma/p\Gamma| = 1$. Then, for any $z \in \dot{F}$, $u(z) = pu(b)$ for some $b \in \dot{F}$. This implies that zb^{-p} is invertible in $A(P)$, hence in A_v. It follows that $v(zb^{-p}) = 0$, and $v(z) = pv(b) \in pG_v$. So we have $G_v = pG_v$; this contradicts condition (3). Therefore $|\Gamma/p\Gamma| = p$.

According to Proposition 2.2, (F, P) is a real closed field of exact level n. This completes the proof. $\square$

Now, we are able to establish the following

Theorem 2.2. *Let (R, P) be an ordered ring of exact level n, and let v be a real valuation on R with valuation pair (A, M), core $P \cap -P$ and value group G. Then, (R, P) is real closed if and only if the following conditions are satisfied*:

(1) *$P \cap -P$ is a prime ideal of $I(R)$, where $I(R)$ is the totally integral closure of R.*
(2) *$R/(P \cap -P)$ is integrally closed in its fraction field.*
(3) *v is henselian.*
(4) *For some divisor m of n, $(A/M, P_v)$ is a real closed ring of exact level m such that $P_v \cap -P_v = \{0\}$, where $P_v = \{a + M \mid a \in P \cap A\}$.*
(5) *For every prime p, $|G/pG| = p$ if $p \mid n$ but $p \nmid m$; otherwise $G = pG$.*

Proof. Put $\wp = P \cap -P$, denote by (F, P_F) the ordered fraction field of $(R/\wp, P/\wp)$, and write v^* the valuation on F induced by v as above. Then, the value group of v^* is still G, and $(A^*_{T^*}, M^*_{T^*})$ is the valuation pair of v^*, where $A^* := A/\wp$, $M^* := M/\wp$, and T^* is the image of $A \setminus M$ under the canonical homomorphism $R \to R/\wp$. Write F_{v^*} for the residue field of v^*. By the argument above, F_{v^*} may be regarded as the fraction field of A/M.

Sufficiency. By condition (4), v is compatible with P. According to Lemma 2, v^* is a henselian valuation compatible with P_F. Write P_{v^*} for the ordering of F_{v^*} induced by P_v. Then (F_{v^*}, P_{v^*}) is obviously the ordered fraction field of $(A/M, P_v)$. Moreover, it is easy to see that P_{v^*} is also induced by P_F. By Proposition 1.2, (F_{v^*}, P_{v^*}) is a real closed field, since $(A/M, P_v)$ is real closed. Observe that G is the value group of v^*. According to Proposition 2.3, (F, P_F) is a real closed field. From Proposition 1.2 and condition (2), it follows that $(R/\wp, P/\wp)$ is a real closed ring. By Theorem 1.2 and condition (1), (R, P) is real closed.

Necessity. Condition (1) follows from Theorem 1.1, and condition (2) follows from Theorem 1.1 and Proposition 1.2. Notice that $A/\wp$ is the intersection of integrally closed subrings $R/\wp$ and $A^*_{T^*}$ inside F. Thus $A/\wp$ is integrally closed in F and is integrally closed in $A^*_{T^*}$. Moreover, by Theorem 1.1 and Proposition 1.1, (F, P_F) is a real closed field of exact level n.

Let $f(x) \in A[x]$ and $\phi(x), \psi(x) \in A/M[x]$ be monic polynomials such that $\lambda(f(x)) = \phi(x)\psi(x)$ and $\phi(x)$, $\psi(x)$ are relatively prime, where λ is the canonical homomorphism of $A[x]$ onto $A/M[x]$. By the isomorphism $A/M \cong (A/\wp)/(M/\wp)$, we may assume $\lambda(f(x)), \phi(x), \psi(x) \in (A/\wp)/(M/\wp)[x] \subseteq F_{v^*}[x]$. By Proposition 2.3, v^* is a henselian valuation. Thereby, $\bar{f}(x) = \Phi(x)\Psi(x)$, where $\bar{f}(x)$ is the image of $f(x)$ under the canonical homomorphism: $A[x] \to A/\wp[x]$, and $\Phi(x)$, $\Psi(x)$ are two relatively prime monic polynomials in $A^*_{T^*}[x]$ such that $\phi(x)$, $\psi(x)$ are respectively the images of $\Phi(x)$, $\Psi(x)$ under the canonical homomorphism: $A^*_{T^*}[x] \to F_{v^*}[x]$. Observe that $A/\wp$ is integrally closed in $A^*_{T^*}$. By a familiar fact about decompositions of monic polynomials (cf. [1, Exercise 6 in Chapter 5]), we have $\Phi(x), \Psi(x) \in A/\wp[x]$. According to Theorem 1 and Corollary 1 in [11], $f(x)$ can be factored in $I(R)[x]$ as follows:

$$f(x) = (x-\alpha_1)\cdots(x-\alpha_n),$$

where $n = \deg f(x)$, and $\alpha_i \in I(R)$, $i = 1, \dots, n$.

Putting $\bar{\alpha}_i = \alpha + \wp \in I(R)/\wp$, $i = 1, \dots, n$, we have

$$\Phi(x)\Psi(x) = \bar{f}(x) = (x-\bar{\alpha}_1)\cdots(x-\bar{\alpha}_n).$$

Since $I(R)/\wp$ is a domain, we may assume $\Phi(x) = (x-\bar{\alpha}_1)\cdots(x-\bar{\alpha}_r)$ and $\Psi(x) = (x-\bar{\alpha}_{r+1})\cdots(x-\bar{\alpha}_n)$. Setting $g(x) = (x-\alpha_1)\cdots(x-\alpha_r)$ and $h(x) = (x-\alpha_{r+1})\cdots(x-\alpha_n)$, we have $f(x) = g(x)h(x)$. Clearly, there exist $g_1(x), h_1(x) \in A[x]$ such that $\Phi(x), \Psi(x)$ are respectively the images of $g_1(x), h_1(x)$ under the canonical homomorphism: $A[x] \to A/\wp[x]$. Then, $g(x) - g_1(x), h(x) - h_1(x) \in \wp[x]$, and $g(x), h(x) \in A[x]$. Obviously, $\phi(x), \psi(x)$ are respectively the images of $g(x), h(x)$ under the canonical homomorphism: $A[x] \to A/M[x]$. This implies that v is a henselian valuation on R.

According to Proposition 2.3, (F_{v^*}, P_{v^*}) is a real closed field of exact level m for some divisor m of n. In order to verify condition (4), it is sufficient to prove that A/M is integrally closed in F_{v^*}.

Assume that $\bar{\alpha}$ is a non-zero element in F_{v^*} and $\bar{\alpha}$ is integral over A/M. Then, for a monic polynomial $\bar{f}(x) \in A/M[x]$, $\bar{f}(\bar{\alpha}) = 0$. Obviously, there is a monic polynomial $f(x)$ in $A/\wp[x]$ such that $\bar{f}(x)$ is the image of $f(x)$ under the canonical homomorphism: $A/\wp[x] \to A/M[x]$. Since $\bar{f}(x)$ is a monic polynomial with root $\bar{\alpha}$, we have $\bar{f}(x) = (x-\bar{\alpha})^m\psi(x)$, where $m \in \mathbb{N}$, and $\psi(x) \in F_{v^*}[x]$ is a monic polynomial such that $\psi(\bar{\alpha}) \neq 0$. Observe that v^* is a henselian valuation on F. Thereby, $f(x) = g(x)h(x)$, where $g(x), h(x)$ are two relatively prime monic polynomials in $A^*_{T^*}[x]$ such that $(x-\bar{\alpha})^m$, $\psi(x)$ are respectively the images of $g(x)$, $h(x)$ under the canonical homomorphism: $A^*_{T^*}[x] \to F_{v^*}[x]$. Since $A/\wp$ is integrally closed in $A^*_{T^*}$, we have $g(x), h(x) \in A/\wp[x]$. Let c is the constant term of $g(x)$. Clearly,

$$c + M/\wp \equiv c + M^*_{T^*} = (-1)^m\bar{\alpha}.$$

Put $f_1(x) = x^m - (-1)^m c$. Then, $\bar{\alpha}$ is a simple root of $\bar{f}_1(x)$ in F_{v^*}, where $\bar{f}_1(x)$ is the images of $f_1(x)$ under the canonical homomorphism: $A/\wp[x] \to A/M[x]$. Repeating the preceding argument for $f_1(x)$, there is a monic and linear polynomial $g_1(x)$ in $A/\wp[x]$ such that $\bar{\alpha}$ is a root of the image of $g_1(x)$ under the canonical homomorphism: $A/\wp[x] \to A/M[x]$. This implies $\bar{\alpha} \in A/M$. Condition (4) is verified.

It remains to verify condition (5). Observe that (F, P_F) is a real closed field of exact level n, G is the value group of v^*, and (F_{v^*}, P_{v^*}) is a real closed field of exact level m. Then, condition (5) follows immediately from Proposition 2.3. This completes the proof. $\square$

In Theorem 2.2, if R is a field, conditions (1) and (2) are automatically satisfied.

As an application of Theorem 2.2, we shall investigate the structure of the orderings of real closed rings. For convenience, for a nonempty subset A of a ring R and $a \in R$, we introduce such a notation $(A : a)$ as follows: $(A : a) := \{z \in R \mid az \in A\}$.

Theorem 2.3. *Let (R, P) be a real closed ring of exact level n. Then P is the only ordering of exact level n of R, and P may be obtained in the way as follows*:

$$P = R^{2n}, \quad \text{if } n \text{ is odd},$$
$$P = R^{2n} \cup -\left(R^{2n} : a\right) \quad \text{for } a \in R^n \setminus R^{2n}, \quad \text{if } n \text{ is even}.$$

Proof. Put $\wp = P \cap -P$, and denote by (F, P_F) the ordered fraction field of $(R/\wp, P/\wp)$. By Theorem 1.1 and Proposition 1.2, (F, P_F) is a real closed field of exact level n, and $R/\wp$ is integrally closed in F. Write v for the trivial valuation on R with valuation pair $(R, \wp)$. Then, v is obviously a real valuation with core $\wp$, and v induces the trivial valuation on F. By Theorem 2.2, v is henselian. For simplicity, we consider only the case when n is even, since the case when n is odd may be treated in the same way. In this case, by Satz 3.7 in [3], we have $P_F = F^{2n} \cup -\alpha F^{2n}$ for every $\alpha \in F^n \setminus F^{2n}$.

Fix one element $a \in R^n \setminus R^{2n}$. By Proposition 1.3, $a \notin \wp$, and $\bar{a} \in (R/\wp)^n \subseteq F^n$. Suppose $\bar{a} \in F^{2n}$. Necessarily $\bar{a} \in (R/\wp)^{2n}$, since $R/\wp$ is integrally closed in F. By the fact that v is henselian, we have $a \in R^{2n}$, a contradiction. Thereby $\bar{a} \in F^n \setminus F^{2n}$. By the preceding argument, $-\bar{a} \in P_F$, and $-a \in P \setminus \wp$. For $z \in -(R^{2n} : a)$, $-az \in R^{2n} \subseteq P$, and $z \in P$. It follows that $R^{2n} \cup -(R^{2n} : a) \subseteq P$.

Conversely, let $z \in P$. If $z \in \wp$, by Proposition 1.3 we have $z \in R^{2n} \cap -R^{2n} \subseteq R^{2n}$. Now assume $z \notin \wp$. By the preceding argument we have $\bar{z} \in P_F = F^{2n} \cup -\bar{a}^{-1} F^{2n}$, since $\bar{a}^{-1} \in F^n \setminus F^{2n}$. Thereby, $\bar{z} \in F^{2n}$ or $\overline{-az} = -\bar{a}\bar{z} \in F^{2n}$. Since $R/\wp$ is integrally closed in F, we have $\bar{z} \in (R/\wp)^{2n}$ or $\overline{-az} \in (R/\wp)^{2n}$. According to the fact that v is henselian, we have $z \in R^{2n}$ or $-az \in R^{2n}$. Hence, $z \in R^{2n} \cup -(R^{2n} : a)$. Therefore, $P \subseteq R^{2n} \cup -(R^{2n} : a)$. This completes the proof. $\square$

Theorem 2.4. *Let (R, P) be a real closed ring of exact level n, and let v be a real valuation on R with valuation pair (A_v, M_v) and value group G_v. Then, one of the following statements is true*:

(1) *n is odd.*
(2) *n is even, and $G_v \neq 2G_v$.*
(3) *n is even, $G_v = 2G_v$, and the fraction field of A_v/M_v is not an euclidean field.*

Proof. Write C for the core of v, denote by (F, P_F) the ordered fraction field of $(R/\wp, P/\wp)$ where $\wp = P \cap -P$, and write K for the fraction field of A_v/M_v.

It is sufficient to prove that K is not an euclidean field if n is even, and $G_v = 2G_v$. By Theorem 2.3, we have $P = R^{2n} \cup -(R^{2n} : b^n)$ for some $b \in R$ with $b^n \in R^n \setminus R^{2n}$. Since $-v(b) \in G_v = 2G_v$, there is an $c \in R$ such that $-v(b) = 2v(c)$. Putting $a := bc^2$. Then $a \in A_v \setminus M_v$. Clearly, $\bar{a}^n \in F^n \setminus F^{2n}$. According to the proof of Theorem 2.3, we have $a^n \in R^n$ and $P = R^{2n} \cup -(R^{2n} : a^n)$. Since A_v is integrally closed in R, we have

$P \cap A_v = A_v^{2n} \cup -(A_v^{2n} : a^n)$. It follows that $P_v = (A_v/M_v)^{2n} \cup -((A_v/M_v)^{2n} : \bar{a}^n)$, where $\bar{a} := a + M_v \in A_v/M_v$. By Theorem 2.1, v is compatible with P. Then P_v is an ordering of A_v/M_v. It is easy to see that $K^{2n} \cup -\bar{a}^{-n} K^{2n}$ is an ordering of K. Putting $P_K := K^{2n} \cup -\bar{a}^{-n} K^{2n}$, we have $|\dot{K}/\dot{P}_K| = 2m$ for some $m \in \mathbb{N}$. Since K is a real field and n is even, we have $\dot{K}^{2n} \cap -\bar{a}^{-n} \dot{K}^{2n} = \emptyset$, and $|\dot{K}/\dot{K}^{2n}| = 4m$. Assume $n = 2^k r$, where r is an odd number. Suppose that K is euclidean. Then, we have $K^{2r} = K^{2n}$. In this case, for every element ξ in the group $\dot{K}^2/\dot{K}^{2n}$, the order of ξ is a divisor of r. Thereby, the order of $\dot{K}^2/\dot{K}^{2n}$ is an odd number, and $|\dot{K}/\dot{K}^2|$ is a multiple of 4. This implies that K is not an euclidean field, a contradiction. Therefore, K is not an euclidean field if n is even, and $G_v = 2G_v$. Theorem 2.4 is proved. □

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